FLUXES IN M-THEORY ON 7-MANIFOLDS: G_2-, SU(3)- AND SU(2)-STRUCTURES

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We consider compactifications of M-theory on 7-manifolds in the presence of 4-form fluxes, which leave at least four supercharges unbroken. Supersymmetric vacua admit G-structures and we discuss the cases of G_2-, SU(3)- as well as SU(2)-structures. We derive the constraints on the fluxes imposed by supersymmetry and determine the flux components that fix the resulting 4-dimensional cosmological constant (i.e. superpotential).

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1. Introduction

An essential input in lifting the continuous moduli space might be non-zero fluxes on the internal space. By now one can find a long list of literature about this subject. A starting point was the work by Candelas and Raine[1] for an un-warped metric which was generalized later in[2] (for an earlier work on warp compactification see [3]) and the first examples, which preserve $\mathcal{N} = 1$ supersymmetry appeared in[4]. The subject was revived around 10 years later by the work of Polchinski and Strominger[5], where flux compactifications in type II string theory was considered. In the M-theory setting, different aspects are discussed in[6,7,8,9,10,11,12,13,14].

Fluxes induces a non-trivial back reaction onto the geometry, because they appear as specific con-torsion components[15,16,17,18,19,9,20,12,21,22] for the Killing spinor. The resulting spaces are in general non-Kählerian, which reflects the fact that the moduli space is (partly) lifted. In order to see which moduli are fixed, one can derive the corresponding superpotential as function of the fluxes in a way discussed in[23], but this approach becomes subtle if the fluxes are not related to closed forms (due to Chern-Simons terms).

In this talk we discuss M-theory compactifications in the presence of 4-form fluxes, which keep the external 4-d space time maximal symmetric, i.e. either flat or anti deSitter (AdS), where in the latter case the superpotential remains non-zero in the vacuum giving rise to a negative cosmological constant. We start by making the Ansatz for the metric and the 4-form field strength and separate the gravitino variation into an internal and external part. In addition, we have to make an Ansatz for the 11-d Killing spinor, which decomposes into internal 7-d spinors and the external 4-d spinors. In the most general case, the solution will be rather involved and we use G-structures to classify possible vacua (Section 3). These structures are defined by a set of invariant differential forms and are in one to one correspondence to the number of internal spinors, which will enter the 11-d Killing spinor. Using these differential forms, one can formely solve the BPS equations (Section 4), but explicit solutions require the construction of these forms. Note, the case of the G_2- and SU(3)-structures have been discussed already before[10,11,12] and we will be rather short.

2. Warp compactification in the presence of fluxes

In the (flux) vacuum, all Kaluza-Klein scalars and vectors are trivial and hence we consider as Ansatz for the metric and the 4-form field strength

$$\begin{aligned} ds^2 &= e^{2A}\big[\, g^{(4)}_{\mu\nu} dx^\mu dx^\nu + \, h_{ab} dy^a dy^b \big] \ , \\ \hat F &= \tfrac{m}{4!}\, \epsilon_{\mu\nu\rho\lambda} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\lambda + \tfrac{1}{4!} F_{abcd}\, dy^a \wedge dy^b \wedge dy^c \wedge dy^d \end{aligned} \tag{1}$$

where $A = A(y)$ is a function of the coordinates of the 7-manifold with the metric h_{ab}, m is the Freud-Rubin parameter and the 4-d metric $g^{(4)}_{\mu\nu}$ is either flat or anti deSitter. Unbroken supersymmetry requires the existence of (at least) one Killing spinor η yielding a vanishing gravitino variation of 11-dimensional supergravity

$$0 = \delta\Psi_M = \big[\partial_M + \tfrac{1}{4}\hat\omega_M^{RS}\Gamma_{RS} + \tfrac{1}{144}\big(\Gamma_M F - 12\, F_M\big)\big]\eta \tag{2}$$

where: $\hat F \equiv F_{MNPQ}\Gamma^{MNPQ}$, $\hat F_M \equiv F_{MNPQ}\Gamma^{NPQ}$, etc. Since,

$$\Gamma_M \Gamma^{N_1\cdots N_n} = \Gamma_M^{\ N_1\cdots N_n} + n\, \delta_M^{[N_1}\Gamma^{N_2\cdots N_n]} \tag{3}$$

one can bring the variation also in the more common form. Using the convention $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$ with $\eta = \mathrm{diag}(-,+,+\ldots+)$, we decompose the Γ-matrices as usual

$$\Gamma^\mu = \hat\gamma^\mu \otimes \mathbb{1} \qquad , \qquad \Gamma^{a+3} = \hat\gamma^5 \otimes \gamma^a \tag{4}$$

with $\mu = 0..3$, $a = 1..7$, and $\hat\gamma^5 = i\hat\gamma^0\hat\gamma^1\hat\gamma^2\hat\gamma^3$, $1 = i\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7$ yields

$$i\hat\gamma^5\hat\gamma^\mu = \frac{1}{3!}\epsilon^{\mu\nu\rho\lambda}\hat\gamma_{\nu\rho\lambda} \quad , \qquad \frac{i}{3!}\epsilon^{abcdmnp}\gamma_{mnp} = \gamma^{abcd} \equiv \gamma^{[a}\gamma^b\gamma^c\gamma^{d]} \ . \tag{5}$$

The spinors in 11-d supergravity are Majorana and we take all 4-d $\hat\gamma^\mu$-matrices are real and $\hat\gamma^5$ as well as the 7-d γ^a-matrices are purely imaginary and antisymmetric.

With this notation, we can now split the gravitino variation into an internal and external part. In order to deal with the warp factor, we use

$$ds^2 = e^{2A}\widetilde{ds}^{\,2} \quad \rightarrow \quad D_M = \tilde D_M + \frac{1}{2}\Gamma_M^{\ N}\partial_N A \tag{6}$$

and find for the external components of the gravitino variation

$$0 = \big[\nabla_\mu \otimes \mathbb{1} + \hat\gamma_\mu\hat\gamma^5 \otimes \big(\frac{1}{2}\,\partial A + \frac{i\,m}{36}\big) + \frac{1}{144}e^{-3A}\,\hat\gamma_\mu \otimes F\big]\eta \tag{7}$$

where $F = F_{abcd}\gamma^{abcd}$, $F_a = F_{abcd}\gamma^{bcd}$, etc. and ∇_μ is the 4-d covariant derivative. In the same way, we get for the internal variation

$$0 = \big[\mathbb{1} \otimes \big(\nabla^{(h)}_a - \frac{1}{2}\partial_a A + \frac{im}{48}\gamma_a\big) - \frac{1}{4}\hat\gamma^5\hat\gamma^\mu\nabla_\mu \otimes \gamma^a - \frac{1}{12}\, e^{-3A}\,\hat\gamma^5 \otimes F_a\big]\eta \tag{8}$$

where we eliminated the term $\sim \gamma_a F\eta$ by using eq. (7).

In order to solve these equations, we have to decompose the spinor and introduce the superpotential yielding the negative cosmological constant. The 11-d Majorana spinor can be expanded in all independent spinors as

$$\eta = \sum_{i=1}^{\mathcal{N}} (\epsilon^i \otimes \theta^i + cc) \ ,$$

where ϵ^i and θ^i denote the 4- and 7-d spinors, resp. If there are no fluxes, all of these spinors are covariantly constant and $\mathcal{N} \leq 8$ gives the resulting extended supersymmetries in 4 dimensions. With non-trivial fluxes one can however impose a relation between the spinors and $\mathcal{N}$ does not refer to the number of unbroken supersymmetries (see last Section), but gives nevertheless a classification of supersymmetric vacua. In fact, with these spinors one can build differential forms that are singlets under a subgroup $G \subset spin(7)$ and hence define a G-structure, where the number of spinors is directly related to the group G (see next Section). By definition, the spinors are singlets under G and therefore obey certain projector conditions, which annihilate all non-singlet components and, at the same time, can be used to derive simple differential equations for the spinors and constraints on the fluxes (see last Section).

If the 4-d spinors are covariantly constant, the resulting vacuum will be a 4-d flat space, but for an anti deSitter vacuum the spinors satisfy

$$\nabla_\mu \epsilon^i \ \sim \ \hat{\gamma}_\mu \, (W_1^{ij} + i\hat{\gamma}^5 \, W_2^{ij}) \, \epsilon_j \ . \tag{9}$$

Note, the resulting 4-d cosmological constant will be: $-|W|^2$ and we did not take into account a Kähler potential, ie. our superpotential will not be holomorphic. If there is only a single spinor this equation simplifies to

$$\nabla_\mu \epsilon \ \sim \ \hat{\gamma}_\mu \, (W_1 + i \, \hat{\gamma}^5 \, W_2) \, \epsilon$$

and if ϵ is a Weyl spinor it becomes $\nabla_\mu \epsilon = \hat{\gamma}_\mu \, \bar{W} \epsilon^\star$ with the complex superpotential $W = W_1 + i \, W_2$. If ϵ_i are a set of Weyl spinors, we introduce the superpotential by a 11-d spinor satisfying the equation

$$\big[\nabla_\mu \otimes \mathbb{1}\big] \eta = (\hat{\gamma}_\mu \otimes \mathbb{1}) \tilde{\eta} \qquad \text{with :} \qquad \tilde{\eta} = W^{ij} \epsilon_i \otimes \theta_j^\star + cc \tag{10}$$

This way of introducing the superpotential might be confusing. Recall, we set constant all 4-d scalars as well as vector potentials and hence the superpotential should just be a number fixing the cosmological constant for the given vacuum. Since we introduced the superpotential in the 11-d Killing spinor equation it will, on the other hand, depend on the fluxes

and the warp factor and thus it is in general not constant over the internal space. The correct 4-dimensional superpotential is of course obtained only after a Kaluza-Klein reduction, i.e. after an integration over the internal space and to make this clear we will denote this constant superpotential by $W^{(0)}$. We do not want to discuss issues related to a concrete Kaluza-Klein reduction (over a not Ricci-flat internal space) and want instead determine the flux components that are responsible for a non-zero value of $W^{(0)}$.

3. *G*-Structures

Supersymmetric compactifications on 7-manifolds imply the existence of differential forms, which are singlets under a group $G \subset spin(7)$ and which define G-structures[a]. These globally defined differential forms can be constructed as bi-linears of the internal Killing spinors as

$$\theta_i \gamma_{a_1 \cdots a_n} \theta_j$$

and the group G is fixed by the number of independent spinors θ_i which are all singlets under G. E.g. if there is only a single spinor on the 7-manifold, it can be chosen as a real G_2 singlet; if there are two spinors, one can combine them into a complex $SU(3)$ singlet; three spinors can be written as $Sp(2) \simeq SO(5)$ singlets and four spinors as $SU(2)$ singlets. Of course, all eight spinors cannot be a singlet of a non-trivial subgroup of $SO(7)$ and G is trivial. The 7-dimensional γ-matrices are in the Majorana representation and satisfy the relation: $(\gamma_{a_1 \cdots a_n})^T = (-)^{\frac{n^2+n}{2}} \gamma_{a_1 \cdots a_n}$, which implies that the differential form is antisymmetric in $[i,j]$ if $n = 1,2,5,6$ and otherwise symmetric [we assumed here of course that θ^i are commuting spinors and the external spinors are anti-commuting]. This gives the well-known statement that having only a single spinor, one cannot build a vector or a 2-form, but only a 3-form and its dual 4-form [the 0- and 7-form exist trivially on any spin manifold]. If we have two spinors $\theta_{\{1/2\}}$, we can build one vector v and one 2-form (and of course its dual 5- and 6-form). Since the spinors are globally well-defined, also the vector field is well defined on X_7 and it can be used to obtain a foliation of the 7-d space by a 6-manifold X_6. Similarly, having three 7-spinors we can build three vector fields as well as three 2-forms and having four spinors the counting yields six vectors combined with six 2-forms. In addition to these vector fields and 2-forms, one obtains further 3-forms the symmetrized combination of the

[a]We follow here basically the procedure initiated in the recent discussion by[17].

fermionic bi-linears. We have however to keep in mind, that all these forms are not independent, since Fierz re-arrangements yield relations between the different forms, see[17,9] for more details.

Using complex notation, we can introduce the following two sets of bi-linears $[\hat{\theta}^\dagger = (\hat{\theta}^*)^T]$:

$$\Omega_{a_1\cdots a_k} \equiv \theta^\dagger \gamma_{a_1\cdots a_k}\theta \qquad \text{and} \qquad \tilde{\Omega}_{a_1\cdots a_k} \equiv \theta^T \gamma_{a_1\cdots a_k}\theta$$

where dropped the index i, j which counts the spinors. The associated k-forms becomes now

$$\Omega^k \equiv \frac{1}{k!}\Omega_{a_1\cdots a_k} e^{a_1\cdots a_k} \qquad \text{and} \qquad \tilde{\Omega}^k \equiv \frac{1}{k!}\tilde{\Omega}_{a_1\cdots a_k} e^{a_1\cdots a_k}\,. \tag{11}$$

If the spinors are covariantly constant (with respect to the Levi-Civita connection) the group G coincides with the holonomy of the manifold. If the spinors are not covariantly constant neither can be these differential forms and the deviation of G from the holonomy group is measured by the intrinsic torsion. In the following we will discuss the different cases in more detail.

G_2 structures

In the simplest case, the Killing spinor is a G_2 singlet and reads

$$\theta = e^Z \theta_0 \tag{12}$$

where θ_0^T is a normalized real spinor. Due to the properties of the 7-d γ-matrices (yielding especially $\theta_0^T\gamma_a\theta_0 = 0$), only the following differential forms are non-zero

$$\begin{aligned} 1 &= \theta_0^T\theta_0 \ , \\ i\,\varphi_{abc} &= \theta_0^T\gamma_{abc}\theta_0 \ , \\ -\psi_{abcd} &= \theta_0^T\gamma_{abcd}\theta_0 \ , \\ i\,\epsilon_{abcdmnp} &= \theta_0^T\gamma_{abcdmnp}\theta_0 \ . \end{aligned} \tag{13}$$

They are G_2-invariant since θ_0 is a G_2 singlet, i.e. it obeys the appropriate projector constraints. Note, the Lie algebra $\mathfrak{so}(7)$ is isomorphic to Λ^2 and a reduction of the structure group on a general X_7 from $SO(7)$ to the subgroup G_2 implies the following splitting:

$$\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{g}_2^\perp\ . \tag{14}$$

This induces a decomposition of the space of 2-forms in the following irreducible G_2-modules,

$$\Lambda^2 \;=\; \Lambda^2_7 \,\oplus\, \Lambda^2_{14}\,, \tag{15}$$

where

$$\Lambda^2_7 = \{u \lrcorner \varphi | u \in TX_7\} = \{\alpha \in \Lambda^2 \,|\, *(\varphi \wedge \alpha) - 2\alpha = 0\}$$
$$\Lambda^2_{14} = \{\alpha \in \Lambda^2 \,|\, *(\varphi \wedge \alpha) + \alpha = 0\} \cong \mathfrak{g}_2$$

with the abbreviation $u \lrcorner \varphi \equiv u^m \varphi_{mnp}$ and φ denotes the G_2-invariant 3-index tensor, which is expressed as fermionic bi-linear in (13). The operator $*(\varphi \wedge \alpha)$ splits the 2-forms correspondingly to the eigenvalues 2 and -1. These relations serve us to define the orthogonal projections $\mathcal{P}_k$ onto the k-dimensional spaces:

$$\mathcal{P}_7(\alpha) = \frac{1}{3}\,(\alpha + *(\varphi \wedge \alpha)) = \frac{1}{3}\,(\alpha + \frac{1}{2}\alpha \lrcorner \psi)\ , \tag{16}$$
$$\mathcal{P}_{14}(\alpha) = \frac{1}{3}\,(2\alpha - *(\varphi \wedge \alpha)) = \frac{2}{3}\,(\alpha - \frac{1}{4}\alpha \lrcorner \psi) \tag{17}$$

where $\psi = *\varphi$. To be concrete, the G_2-singlet spinor satisfies the condition

$$(\mathcal{P}_{14})^{cd}_{\ \ ab}\,\gamma_{cd}\,\theta_0 = \frac{2}{3}\big(\mathbb{1}^{cd}_{\ \ ab} - \frac{1}{4}\psi^{cd}_{\ \ ab}\big)\gamma_{cd}\,\theta_0 = 0$$

which is equivalent to

$$\begin{aligned} \gamma_{ab}\theta_0 &= i\varphi_{abc}\gamma^c\theta_0\ , \\ \gamma_{abc}\theta_0 &= \big(i\varphi_{abc} + \psi_{abcd}\gamma^d\big)\,\theta_0\ , \\ \gamma_{abcd}\theta_0 &= \big(-\psi_{abcd} - 4i\varphi_{[abc}\gamma_{d]}\big)\theta_0 \end{aligned} \tag{18}$$

where the second and third conditions follow from the first one. These relations can now be used to re-cast the Killing spinor equations into constraints for the fluxes and differential equations for the warp factor as well as the spinor θ. In the generic situation this spinor is not covariantly constant, which reflects the fact that fluxes deform the geometry by the gravitational back reaction. This can be made explicit by re-writing the flux terms as con-torsion terms[b]

$$\tilde{\nabla}_a\theta \equiv (\nabla_a - \frac{1}{4}\tau_a^{\ bc}\gamma_{bc})\theta = 0\ .$$

[b]There is also an ongoing discussion in the mathematical literature, see[24].

From the symmetry it follows that τ has $7\times 21 = 7\times(7+14)$ components, but if θ is a G_2-singlet the **14** drops out and hence $\tau \in \Lambda^1\otimes \mathfrak{g}_2^{\perp}$. These components decompose under G_2 as

$$\mathbf{49} = \mathbf{1}+\mathbf{7}+\mathbf{14}+\mathbf{27} = \tau_1+\tau_7+\tau_{14}+\tau_{27}$$

where τ_i are called G_2-structures. Since the Killing spinors define φ and ψ, these torsion classes can be obtained from $d\varphi$ and $d\psi$ as follows

$$\begin{aligned} d\varphi \in \Lambda^4 \;&=\; \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \ , \\ d\psi \in \Lambda^5 \;&=\; \Lambda^5_7 \oplus \Lambda^5_{14} \ , \end{aligned} \tag{19}$$

where the **7** in Λ^4_7 is the same as in Λ^5_7 up to a multiple. For a general 4-form β, the different projections are

$$\begin{aligned} \mathcal{P}_1(\beta) &= \tfrac{1}{4!}\psi \lrcorner \beta \ , \\ \mathcal{P}^4_7(\beta) &= -\tfrac{1}{3!}\varphi \lrcorner \beta \ , \\ \mathcal{P}_{27}(\beta)_{ab} &= \tfrac{1}{3!}(\beta_{cde\{a}\psi_{b\}}{}^{cde})_0 \end{aligned} \tag{20}$$

where in $(\cdot)_0$ we removed the trace. Thus, the different components in the differentials $d\varphi$ can be obtained from

$$\begin{array}{ll} \tau^{(1)} \longleftrightarrow \psi \lrcorner d\varphi \quad , & \tau^{(7)} \longleftrightarrow \varphi \lrcorner d\varphi \ , \\ \tau^{(14)} \longleftrightarrow *d\psi - \tfrac{1}{4}(*d\psi)\lrcorner \psi \ , & \tau^{(27)} \longleftrightarrow (d\varphi_{cde\{a}\psi_{b\}}{}^{cde})_0 \ , \end{array} \tag{21}$$

where τ_{14} and τ_{27} have to satisfy: $\varphi_3\wedge\Lambda^3_{27} = \varphi_3\wedge\tau_{14} = 0$.

SU(3) structures

Having a $G = SU(3)$, one can find two singlet spinors on X_7, which are equivalent to the existence of a vector field v. This in turn can be used to combine both spinors into one complex spinor defined as

$$\theta = \frac{1}{\sqrt{2}}\, e^Z\,(\mathbb{1}+v_a\gamma^a)\theta_0 \quad , \qquad v_a v^a = 1 \tag{22}$$

where the constant spinor θ_0 is again the G_2 singlet and Z is now a complex function. The vector v is globally well-defined and gives a foliation of X_7 by a 6-manifold X_6 and both spinors, θ and its complex conjugate $\theta^\star$, are chiral spinors on X_6. In this case, we have to distinguish between the forms

Ω and $\tilde{\Omega}$ as defined in (11) and find[11,10,12]

$$
\begin{aligned}
\Omega^0 &= e^{2\,\mathrm{Re}(Z)}\ , \\
\Omega^1 &= e^{2\,\mathrm{Re}(Z)}\, v\ , \\
\Omega^2 &= i\, e^{2\,\mathrm{Re}(Z)}\, v \lrcorner \varphi \,=\, i\, e^{2\,\mathrm{Re}(Z)}\, \omega\ , \\
\Omega^3 &= i\, e^{2\,\mathrm{Re}(Z)} \big[v \wedge (v \lrcorner \varphi)\big] \,=\, i\, e^{2\,\mathrm{Re}(Z)}\, v \wedge \omega\ , \\
\tilde{\Omega}^3 &= i\, e^{2\,\mathrm{Re}(Z)} \big[e^{2i\,\mathrm{Im}(Z)} \big(\varphi \,-\, v \wedge \omega \,-\, i\, v \lrcorner \psi\big)\big] \,=\, i\, e^{2\,\mathrm{Re}(Z)}\, \Omega^{(3,0)}
\end{aligned}
\tag{23}
$$

and all other forms are zero or dual to these ones. The associated 2-form to the almost complex structure on X_6 is ω and with the projectors $\frac{1}{2}(\mathbb{1} \pm i\omega)$ we can introduce (anti) holomorphic indices[c] so that $\Omega^{(3,0)}$ can be identified as the holomorphic $(3,0)$-form on X_6. There exists a topological reduction from a G_2-structure to a $SU(3)$-structure (even to a $SU(2)$-structure). The difficulties arise by formulating the geometrical reduction. Using the vector v, let the explicit embedding of the given $SU(3)$-structure in the G_2-structure be:

$$
\begin{aligned}
\varphi &= \mathrm{Re}(e^{-2i\,\mathrm{Im}(Z)}\, \Omega^{(3,0)}) + v \wedge \omega \,=\, \chi_+ +\, v \wedge \omega\ , \\
\psi &= \mathrm{Im}(e^{-2i\,\mathrm{Im}(Z)}\, \Omega^{(3,0)}) \wedge v \,+\, \tfrac{1}{2}\omega^2 \,=\, \chi_- \wedge v \,+\, \tfrac{1}{2}\omega^2
\end{aligned}
\tag{24}
$$

with the compatibility relations

$$
e^{-2i\,\mathrm{Im}(Z)}\, \Omega^{(3,0)} \wedge \omega = (\chi_+ \,+\, i\,\chi_-) \wedge \omega \,=\, 0\,, \tag{25}
$$

$$
\chi_+ \,\wedge\, \chi_- = \frac{2}{3}\, \omega^3\ . \tag{26}
$$

Now, the projectors (18) for θ_0 imply for the complex 7-d in (22)

$$
\begin{aligned}
\gamma_a \theta &= \tfrac{e^Z}{\sqrt{2}} (\gamma_a + v_a + i\varphi_{abc} v^b \gamma^c)\theta_0\ , \\
\gamma_{ab} \theta &= \tfrac{e^Z}{\sqrt{2}} (i\varphi_{abc}\gamma^c + i\varphi_{abc} v^c + \psi_{abcd} v^c \gamma^d - 2 v_{[a}\gamma_{b]})\theta_0\ , \\
\gamma_{abc} \theta &= \tfrac{e^Z}{\sqrt{2}} (i\varphi_{abc} + \psi_{abcd}\gamma^d + 3i v_{[a}\varphi_{bc]d}\gamma^d - \psi_{abcd} v^d - 4i\varphi_{[abc}\gamma_{d]} v^d)\theta_0\,, \\
\gamma_{abcd} \theta &= \tfrac{e^Z}{\sqrt{2}} (-\psi_{abcd} - 4i\varphi_{[abc}\gamma_{d]} - 5\psi_{[abcd}\gamma_{e]} v^e \\
&\qquad -4i v_{[a}\varphi_{bcd]} - 4 v_{[a}\psi_{bcd]e}\gamma^e)\theta_0\ , \\
\gamma_{abcde} \theta &= \tfrac{e^Z}{\sqrt{2}} (-5\psi_{[abcd}\gamma_{e]} - i\varepsilon_{abcdefg}\gamma^g v^f - 5 v_{[a}\psi_{bcde]} - 20 i v_{[a}\varphi_{bcd}\gamma_{e]})\theta_0\,, \\
\gamma_{abcdef} \theta &= \tfrac{e^Z}{\sqrt{2}} (-i\varepsilon_{abcdefg}\gamma^g + \varepsilon_{abcdefg} v_h \gamma_j \varphi^{ghj} - i\varepsilon_{abcdefg} v^g)\,\theta_0\ .
\end{aligned}
$$

[c]Since the 6-d space is in general not a complex manifold, we cannot introduce global holomorphic quantities and this projection is justified only pointwise.

Again, these relations can be used to re-write the Killing spinor equations in terms of constraint equations for the fluxes and a differential equation for the warp factor as well as the spinor. The corresponding torsion components[25] are now related to the differential equation obeyed by the forms: v, ω, Ω and their dual.

As next case one would consider SP(2) structures implying three (real) singlet spinors. An example is a 7-d 3-Sasaki-space (i.e. the cone yields an 8-d Hyperkähler space with Sp(2) holonomy), with the Aloff-Walach space $N^{1,1}$ as the only regular examples[26] (apart from S^7); non-regular examples are in[27]. We leave a detailed discussion of this case for the future and investigate instead the SU(2) case in more detail.

SU(2) structures

On any 7-d spin manifold exist three no-where vanishing vector fields[28], which implies that one can always define SU(2) structures. The corresponding four (real) spinors can be combined in two complex SU(2) singlet spinors $\theta_{1/2}$. The three vector fields v_α, $\alpha = 1, 2, 3$ can be chosen as

$$v_1 = e^1 \qquad v_2 = e^2 \qquad v_3 = \varphi(v_1, v_2)$$

and they parameterize a fibration over a 4-d base space X_4. The embedding of the SU(2) into the G_2 structures is then given by

$$\varphi = v_1 \wedge v_2 \wedge v_3 + v_\alpha \wedge \omega_\alpha \ , \tag{27}$$

$$\psi = vol_4 + \epsilon^{\alpha\beta\gamma} v_\alpha \wedge v_\beta \wedge \omega_\gamma \ . \tag{28}$$

Since the vector fields are no-where vanishing, we can choose them of unit norm and perpendicular to each other, i.e. $(v_\alpha, v_\beta) = \delta_{\alpha\beta}$, and using the 3-form φ, one obtains a cross product of these vectors. One can pick one of these vectors, say v_3, to define a foliation by a 6-manifold and on this 6-manifold one can introduce an almost complex structure by $J = v_3 \lrcorner \varphi \ \in T^*M^6 \otimes TM^6$. The remaining two vectors, which can be combined into a holomorphic vector[d] $v_1 + i\, v_2$ imply that this 6-manifold is a fibration over the base X_4. On this 4-manifold we can define a basis of anti-selfdual 2-forms whose pullback correspond to the ω_α. Note, on any general 4-d manifold we have the splitting

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$$

[d]Meaning, that it is annihilated by the projector: $(\mathbb{1} - J)$.

where we can take $\{\omega_1, \omega_2, \omega_3\}$ as a basis of Λ^2_- and this splitting appears in group theory as: $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The 2-forms satisfy the algebraic relations

$$\omega_i^2 = 2\, vol_4 \qquad \omega_i \wedge \omega_j = 0 \quad \text{for}\, i \neq j$$

and the associating complex structures fulfill the quaternionic algebra (note: the orientation on the 4-fold is negative). We can further split the 2-forms into a symplectic 2-form, say $\omega = \omega_3$, and the remaining can be combined into complex (2,0)-form. Thus, the subbundle Λ^2_- decomposes as

$$\Lambda^2_- \cong \lambda^{2,0} \oplus \mathbb{R}\, \omega$$

and besides the symplectic form ω, we introduce the complexified 2-form:

$$\lambda = \omega_1 + i\, \omega_2$$

which is, with respect to ω, a holomorphic (2,0)-form (due to the quaternionic algebra satisfied by these forms). The SU(2) singlet spinors can again be constructed from the G_2 singlet spinor θ_0 by

$$\theta_1 = \frac{1}{\sqrt{2}}(1 + v_3)\theta_0 \quad , \qquad \theta_2 = v_1\, \theta_1 \tag{29}$$

where $v_\alpha \equiv v_\alpha^m \gamma_m$. With the relations (18), it is straightforward to verify that: $(v_1 v_2 - i v_3)\theta_0 = 0$ and hence

$$(v_1 - i v_2)\theta_2 = (v_1 + i v_2)\theta_1 = 0 \qquad \text{or}: \quad v_\alpha (\sigma^\alpha)_k{}^l \theta_l = \theta_k \ .$$

Moreover,

$$\begin{aligned} v_\alpha v_\beta \theta_k &= \delta_{\alpha\beta}\theta_k + i\epsilon_{\alpha\beta\lambda}(\sigma^\lambda)_k{}^l \theta_l \ , \\ \hat{\omega}\, \theta_k &= 4i\, \theta_k \ , \\ \hat{\lambda}\, \theta_k &= 8\, (\sigma_2)_k{}^l \theta_l^\star \end{aligned} \tag{30}$$

where $\hat{\omega} \equiv \omega_{mn}\gamma^{mn}$, $\hat{\lambda} \equiv \lambda_{mn}\gamma^{mn}$ and with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{31}$$

For the forms (11) we find

$$\begin{aligned}
&\Omega^{(0)} = \mathbb{1} \ , && \tilde\Omega^{(0)} = 0 \ , \\
&\Omega^{(1)} = v^\alpha \sigma_\alpha \ , && \tilde\Omega^{(1)} = 0 \ , \\
&\Omega^{(2)} = i\,\omega\,\mathbb{1} + \Omega^{(1)} \wedge \Omega^{(1)} \ , && \tilde\Omega^{(2)} = -\lambda^\star\,\sigma_2 \ , \\
&\Omega^{(3)} = \Omega^{(1)} \wedge \Omega^{(2)} \ , && \tilde\Omega^{(3)} = -\Omega^{(1)} \wedge \tilde\Omega^{(2)} \ , \\
&\Omega^{(4)} = i\,\Omega^{(1)} \wedge \Omega^{(1)} \wedge \omega - vol_4\,\mathbb{1} \ , && \tilde\Omega^{(4)} = -\Omega^{(1)} \wedge \tilde\Omega^{(3)} \ , \\
&\Omega^{(5)} = \Omega^{(1)} \wedge \Omega^{(4)} \ , && \tilde\Omega^{(5)} = \Omega^{(1)} \wedge \Omega^{(1)} \wedge \Omega^{(1)} \wedge \lambda\sigma_2 .
\end{aligned} \tag{32}$$

4. BPS constraints

Now we can come back to the BPS equations from Section 2. With the superpotential as introduced before, equation (7) becomes

$$0 = \tilde\eta + \big[\hat\gamma^5 \otimes \big(\frac{1}{2}\partial A + \frac{im}{36}\big) + \frac{1}{144} e^{-3A}\,(\mathbb{1}\otimes F)\big]\eta \tag{33}$$

and if: $\hat\eta = e^{-\frac{A}{2}}\eta$, equation (8) yields

$$0 = \mathbb{1} \otimes \big(\nabla^{(h)}_a + \frac{i\,m}{48}\,\gamma_a\big)\hat\eta - i\,\hat\gamma^5\gamma_a\,e^{-\frac{A}{2}}\,\tilde\eta - \frac{1}{12}\,e^{-3A}\,\hat\gamma^5 \otimes F_a\hat\eta \ . \tag{34}$$

It is useful to decompose the 35 components of the 4-form field strength under G_2 as $\mathbf{35} \to \mathbf{1} + \mathbf{7} + \mathbf{27}$ with

$$F_{abcd} = \frac{1}{7}\,\mathcal{F}^{(1)}\,\psi_{abcd} + \mathcal{F}^{(7)}_{[a}\,\varphi_{bcd]} - 2\,\mathcal{F}^{(27)}_{e[a}\,\psi^e{}_{bcd]} \tag{35}$$

where $\mathcal{F}^{(1)}$, $\mathcal{F}^{(7)}$ and $\mathcal{F}^{(27)}$ are the projection introduced in (20). The cases of G_2 and SU(3) structures have been discussed already in the literature and we will summarize only the main results.

G_2 structure

In this case, the 11-d spinor is a direct product, i.e.

$$\eta = \epsilon \otimes \theta \tag{36}$$

and since the 11- and 7-d spinor are Majorana also the 4-d spinor ϵ has to be Majorana (a more detailed discussion is given in[29]). One finds that all internal 4-form components have to vanish

$$F_{abcd} = 0 \quad , \qquad W_1 = 0 \ , \quad m = -36 W_2 \ . \tag{37}$$

The equation (34) gives a differential equations for the spinor $e^Z\theta_0$, which implies

$$\partial_a Z = 0 \ .$$

The differential equations for θ_0 fixes the 7-manifold to have a weak G_2 holonomy and hence is Einstein with the cosmological constant given by the Freud-Rubin parameter[29,10]. This in turn implies, that the 8-d space built as a cone over this 7-manifold has $Spin(7)$ holonomy. In fact, after taking into account the vielbeine, this gives the known set of first order differential equations for the spin connection 1-form ω^{ab}: $\omega^{ab}\varphi_{abc} = \frac{7}{36}\, m\, e_b$, where m was the Freud-Rubin parameter [note ω is here the spin connection and should not be confused with the associated 2-form introduced before]. Using the differential equation for the 7-spinor, it is straightforward to verify that

$$d\varphi = -\frac{7\,m}{18}\,\psi \quad , \qquad d\psi = 0$$

and therefore only $\tau^{(1)}$ is non-zero.

The 4-d superpotential is only given by the Freud-Rubin parameter, ie.

$$W^{(0)} \;\sim\; i\int_{X_7} {}^{\star}F \tag{38}$$

which fixes the overall size of the 7-manifold. In the limit of flat 4-d Minkowski vacuum, the Freud-Rubin parameter has to vanish and we get back to the Ricci-flat G_2-holonomy manifold. In order to allow for non-trivial fluxes one has to consider SU(3) instead of G_2 structures.

SU(3) structure

In this case, there is one (complex) 7-d spinor and the 11-d Majorana spinor reads

$$\eta = \epsilon \otimes \theta + \epsilon^{\star} \otimes \theta^{\star} \ . \tag{39}$$

where the 4-d spinors ϵ and $\epsilon^{\star}$ have opposite chirality ($\gamma^5\epsilon = \epsilon$). More details about this case can be found in[11,10]. The solution of (33) read now

$$\begin{aligned} W = W_1 + i\,W_2 &= \tfrac{1}{6}e^{-(K/2+3A)}\left[\tfrac{4}{7}\mathcal{F}^{(1)} - v^a\mathcal{F}^{(27)}_{ab}v^b + iv^a\mathcal{F}^{(7)}_a\right] \\ v^a\partial_a e^{3A} &= \tfrac{3}{7}\mathcal{F}^{(1)} + v^a\mathcal{F}^{(27)}_{ab}v^b \\ m &= 0\ . \end{aligned} \tag{40}$$

and

$$(\delta_a{}^b - v_a v^b)\mathcal{F}^{(7)}_b = \varphi_{abc}v^b\partial^c e^{3A} \;=\; 2\,\varphi_a{}^{bc}v_b\,\mathcal{F}^{(27)}_{cd}v^d\ , \tag{41}$$

$$2\,\mathcal{F}^{(27)}_{ab}v^b = \Big[-\frac{3}{7}\mathcal{F}^{(1)} + v^c\mathcal{F}^{(27)}_{cb}v^b\Big]v_a + \partial_a e^{3A} \tag{42}$$

[the flux components were introduced in (35)]. In addition, one obtains a differential equation for the spinor with the non-trivial torsion components as introduced in (19)

$$\begin{aligned} &\tau^{(1)} \longleftrightarrow W_2 \ , \\ &\tau_a^{(7)} \ \longleftrightarrow\ 48\, W_1\, v_a - \tfrac{24}{7}\, \mathcal{F}^{(1)}\, v_a + \tfrac{3}{2}\, \varphi_a{}^{bc} v_b \mathcal{F}_c^{(7)} + 27 \mathcal{F}_{ab}^{(27)} v^b \ . \end{aligned} \tag{43}$$

To make the set of equations complete, we have to give the differential equations obeyed by the vector field v, which is straightforward if we use the differential equation for the spinor

$$\begin{aligned} \nabla_m v_n &= -\tfrac{1}{12} e^{-3A-2\mathrm{Re}(Z)}\, \theta^\dagger [F_m, \gamma_n] \theta \\ &= \tfrac{1}{12} e^{-3A} F_{mbcd} \omega^{bc} \omega^d{}_n \end{aligned} \tag{44}$$

recall $\omega_{ab} = \varphi_{abc} v^c$. Note, $v^n \nabla_m v_n = 0$, which is consistent with $|v|^2 = 1$. Using the decomposition (35) one finds

$$\nabla_{[m} v_{n]} = \big(\delta_{[m}{}^a \delta_{n]}{}^b + \frac{1}{2} \psi_{mn}{}^{ab}\big) \mathcal{F}_{ac}^{(27)} v^c v_b + \frac{1}{4} \varphi_{mn}{}^a (\delta_a{}^b - v_a v^b) \mathcal{F}_b^{(7)} \,, \tag{45}$$

$$\begin{aligned} \nabla_{\{m} v_{n\}} = &-\frac{2}{7} (\delta_{mn} - v_{\{m} v_{n\}})\, \mathcal{F}^{(1)} - \frac{1}{2} v_{\{m} \varphi_{n\}}{}^{ab} v_a \mathcal{F}_b^{(7)} \\ &+ \frac{1}{2} \big(\delta_m{}^a \delta_n{}^b + \omega_m{}^a \omega_n{}^b\big) \mathcal{F}_{ab}^{(27)} - \frac{1}{2} \delta_{mn} \mathcal{F}_{ab}^{(27)} v^a v^b \ . \end{aligned} \tag{46}$$

The first term in the anti-symmetric part is the projector onto the **7**, see (16), and by contracting with φ and employing eqs. (41) and (42), one can verify that[11]: $d(e^{3A} v) = 0$. One can project the flux components onto X_6 and using the symplectic 2-form ω we can introduce (anti) holomorphic indices. As result, we can define a 3-form H and 4-form G on X_6 and find for the superpotential

$$W = \frac{i}{36} \bar{\Omega}^{(0,3)} \lrcorner H \qquad \rightarrow \qquad W^{(0)} \sim \frac{1}{36} \int_{X_7} F \wedge \Omega^{(3,0)} \tag{47}$$

whereas the 4-form has to fulfill the constraint: $\Omega \lrcorner G = 0$ and $de^{3A} \lrcorner \omega = \frac{1}{2} \omega \lrcorner H$ as well as $v \lrcorner de^{3A} = \frac{1}{144} \omega^2 \lrcorner G$.

SU(2) structure

Finally, in the SU(2) case we write the 11-d spinor as

$$\eta = \epsilon^1 \otimes \theta_1 + \epsilon^2 \otimes \theta_2 + cc \tag{48}$$

and we choose chiral 4-d spinors with

$$\hat{\gamma}^5 \epsilon^i = \epsilon^i \ .$$

Eq. (33) gives

$$0 = \epsilon^i \otimes \Big[W_i{}^j \theta_j^\star + (\frac{1}{2}\partial A + \frac{im}{36} + \frac{1}{144} e^{-3A} F)\,\theta_i \Big] \ . \tag{49}$$

If one does not impose any constraints on the spinors ϵ^i, one finds[14]

$$W_{ij} \sim \theta_i F \theta_j = F \lrcorner \, \tilde{\Omega}^{(4)}_{ij}$$

with the 4-form $\tilde{\Omega}^{(4)}$ as derived in (32). Defining the 2-forms:

$$G_{\alpha\beta} = v^m_\alpha v^n_\beta F_{mnab} \lambda^{ab} \quad , \qquad F_{\alpha\beta} = v^m_\alpha v^n_\beta F_{mnab} \omega^{ab}$$

we can write W_{ij} as matrix: $W \sim (\epsilon^{\alpha\beta\gamma} G_{\alpha\beta}\sigma_\gamma)\sigma_2$ with the σ_α as Pauli matrices. It would be identical zero if: $G = 0$, but instead we can also impose: $\epsilon^i W_{ij} = 0$ so that W_{ij} projects out one of the 4-d spinor as we would need for an $\mathcal{N} = 1$ vacuum. This implies that: $\det W = 0$ which gives one (complex) constraint on the 2-form G. As next step, the contraction with $\theta_k^\dagger$ yields

$$m = 0 \quad , \qquad dA \lrcorner \, \Omega^{(1)} \sim F \lrcorner \, \Omega^{(4)}$$

which implies that: $\partial_\alpha A \sim \epsilon_{\alpha\beta\gamma} F^{\beta\gamma}$ (with $\partial_\alpha \equiv v^m_\alpha \partial_m$) and $F \lrcorner \, vol_4 = 0$. Finally, one has to contract with $\theta\gamma_a$ as well as with $\theta^\dagger\gamma_a$ (with the index a projected onto the base) and if we assume that the $\partial_b A = 0$ (ie. the warp factor is constant over the 4-d base), we get as further contraints on the fluxes

$$\theta^\dagger \gamma_a F \theta = 0 \quad , \qquad \theta \gamma_a F \theta = 0 \ .$$

These constraints are solved, e.g., if the only non-zero components of F are: $\sim v_\alpha \wedge v_\beta \wedge \omega$; ie. are contained in $F_{\alpha\beta}$ and $G_{\alpha\beta} = 0$ (as defined above).

These are all constraints on the fluxes, but from the internal variation (34) we get differential equations. Setting, $m = 0$ and $\tilde{\eta} = 0$, we find

$$\nabla_m \theta_i \ \sim \ F_{mnpq} \gamma^{npq} \theta_i$$

If only the components in $F_{\alpha\beta}$ are non-zero, it is straightforward to further simplify this equation by using the relations in (30). On the other hand, this equation fixes also the corresponding differential equations obeyed by the differential forms.

$$\nabla_k \Omega^{(n)}_{m_1 m_2 \ldots} \sim F_k{}^{npq}\, \theta^\dagger [\gamma_{npq}, \gamma_{m_1 m_2 \ldots}] \theta \ .$$

For the 2-forms eg., our constraints on the fluxes imply that ω and λ are closed, when projected onto the 4-d base, which is therefore a hyper Kähler space. Unfortunately, we have to leave a detailed analysis of these equaions for the future.

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